Advanced Short-Term Actuarial Mathematics Exam

Exam ASTAM

Date: April 24, 2024

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 6 questions numbered 1 through 6 with a total of 60 points. The points for each question are indicated at the beginning of the question.

2. Question 1 is to be answered in the Excel workbook. For this question only the work in the Excel workbook will be graded.

3. Questions 2-6 are to be answered in pen in the Yellow Answer Booklet provided. For these questions graders will only look at the work in the Yellow Answer Booklet. Excel may be used for calculations or for statistical functions, but any work in the Excel booklet will not be graded.

4. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

Excel Answer Instructions

1. For Question 1, you should answer directly in the Excel Question worksheet. The question will indicate where to record your answers.
2. You should generally use formulas in Excel rather than entering solutions as hard coded numbers. This will aid graders in assigning appropriate credit for your work.
3. Graders for Question 1 will not have access to any comments or calculations provided in the Yellow Answer Booklet.
4. For Question 1, you may add notes to the Excel Question worksheet if you feel that might help graders. However, these should be entered directly into the Excel Question worksheet. Graders may not be able to read notes entered as comments.
5. You may use any part of the Excel Question worksheet for additional calculations.

Pen and Paper Answer Instructions

1. Write your candidate number and the number of the question you are answering at the top of each sheet. Your name must not appear.

2. Start each question on a fresh sheet. You do not need to start each sub-part of a question on a new sheet.

3. Write in pen on the lined side of the answer sheet.

4. The answer should be confined to the question as set.

5. When you are asked to calculate, show all your work including any applicable formulas in the Yellow Answer Booklet.

6. If you use Excel for calculations for pen and paper answers, you should include as much information in the Yellow Answer Booklet as if you had used a calculator, including formulas and intermediate calculations where relevant. Written answers without sufficient support will not receive full credit.

7. When you finish, hand in <u>all</u> your written answer sheets to the Prometric Center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

BEGINNING OF EXAMINATION
ADVANCED SHORT-TERM ACTUARIAL MATHEMATICS

Provide the response for Question 1 in the Excel Question worksheet

1.

(*9 points*) You are given the following triangle of the number of cumulative settled claims for 5 Accident Years (AYs) for a particular line of business. Assume that claims are fully settled by the end of Development Year (DY) 4.

Cumulative Number of Claims Settled

AY i	DY, j 0	1	2	3	4
0	49	120	147	166	172
1	46	105	145	172	
2	56	119	151		
3	43	101			
4	36				

(a) (*3 points*)

(i) Estimate the projected cumulative settled claims through each outstanding DY, for AY 1, AY 2, AY 3, and AY 4, using the Chain Ladder method.

(ii) Calculate the estimated number of (incremental) claims settled in each outstanding DY, for AY 1, AY 2, AY 3, and AY 4.

1. Continued

You are also given the following triangle of incremental claims paid for each AY i and DY j where $i + j \leq 4$.

Incremental Amount of Claims Paid					
AY	DY, j				
i	0	1	2	3	4
0	17,558	32,917	39,120	30,464	15,628
1	16,866	28,923	47,297	34,091	
2	20,998	31,429	40,631		
3	18,575	19,134			
4	21,476				

(b) (*3 points*) Calculate the average incremental cost per settled claim for each development year.

(c) (*2 points*) Estimate the aggregate outstanding claims, using the frequency-severity approach and your answers to (a) and (b).

(d) (*1 point*) Explain why claims settled in later Development Years tend to be higher than claims settled in earlier Development Years.

2.

(*8 points*) XYZ Insurance sells medical liability insurance.

(a) (*1 point*) Briefly describe two differences between the distributions of the loss for a single policy and the average loss per policy for a large portfolio of independent policies.

You are the ratemaking actuary for XYZ Insurance. XYZ uses the following data to determine the indicated premium rate on its medical liability insurance:

Expected Effective Period Incurred Losses (000s)	25,000
Fixed Expenses (000s)	3,000
Earned Exposure Units (000s)	20
Permissible Loss Ratio	0.85

(b) (*2 points*) Show that the indicated premium rate is 1,650 to the nearest 10. You should calculate the value to the nearest 1.

XYZ sells medical insurance in Regions A, B, and C. XYZ used the following differentials and had the following experience:

Region	**Existing Differential**	**Experience Period Loss Ratio at Current Rates**	**Exposure Units (000s)**
A	0.80	0.72	5
B	1.00	0.80	10
C	1.20	0.76	5

(c) (*1 point*) Determine the indicated differential for each region using the loss ratio method.

(d) (*2 points*) Another method for calculating new differentials is the loss cost method.

(i) Explain one reason why the differentials calculated by the two methods might produce different results.

(ii) Explain what conditions are required for the two methods to produce the same results.

2. **Continued**

You are given the following additional data:

Base Premium Rate for Region B	1,500
Target Premium Increase	9.8%

(e) (*2 points*) Calculate the revised base premium rate such that the premium change and the differentials are balanced.

3.

(*10 points*) Let S denote the aggregate claims for a portfolio of policies issued by ABC insurance company. You are given:

(i) The claim frequency, N, belongs to the $(a,b,0)$ class with $\mathrm{E}[N]=1.0$ and $\mathrm{Var}[N]=0.9$

(ii) The claim severity, X, also belongs to the $(a,b,0)$ class with $\mathrm{E}[X]=10.0$ and $\mathrm{Var}[X]=60.0$

(a) (*2 points*) Calculate the mean and standard deviation of S.

(b) (*2 points*)

(i) Explain why N must have a binomial distribution.

(ii) Identify the parameters of the distribution of N.

(iii) Identify the distribution and parameters of X.

(c) (*1 point*) Determine the probability generating function (pgf) of S.

(d) (*3 points*) Using the pgf or otherwise, calculate the following probabilities:

(i) $\Pr[S=0]$

(ii) $\Pr[S>1]$

(e) (*2 points*) ABC insurance is entering a stop-loss reinsurance agreement with a deductible $d=2$. Calculate the net stop-loss premium.

4.

(*10 points*) The loss severity for an insurance policy has the following density function:

$$f(x) = \frac{3x^2 e^{-(x/\theta)^3}}{\theta^3} \qquad \text{for } x > 0$$

For this distribution, the first 3 raw moments are:

$$\mathrm{E}\left[X\right] = \theta\Gamma\left(1+\frac{1}{3}\right) \quad \mathrm{E}\left[X^2\right] = \theta^2\,\Gamma\left(1+\frac{2}{3}\right) \quad \mathrm{E}\left[X^3\right] = \theta^3$$

(Note: You can use the GAMMA function in Excel.)
A random sample of 100 losses, $x_1, x_2, \cdots, x_{100}$, has been collected. You are given:

$$\sum_{i=1}^{100} x_i = 954 \quad \sum_{i=1}^{100} x_i^2 = 9{,}888 \quad \sum_{i=1}^{100} x_i^3 = 109{,}812$$

(a) (*2 points*) Determine the log-likelihood function for this sample.

(b) (*2 points*) Show that the maximum likelihood estimate of θ, denoted $\hat{\theta}$, is 10.3 to the nearest 0.1. You should calculate the value to the nearest 0.0001.

(c) (*2 points*) Use Fisher's information to estimate the standard deviation of $\hat{\theta}$.

(d) (*2 points*)

(i) Calculate the maximum likelihood estimate of $\mathrm{Var}\left[X\right]$.

(ii) Using the delta method, estimate a 95% confidence interval for $\mathrm{Var}\left[X\right]$.

(e) (*2 points*) Maximum likelihood estimates are known to have the following characteristics:

A. Invariance property

B. Asymptotically normally distributed

(i) Briefly describe each of these characteristics.

(ii) Explain where each of these characteristics has been applied in this question.

5.

(*12 points*) A company sells three classes of policies, A, B, and C, in equal numbers. The per-policy aggregate loss distributions for policies in each class, for one exposure period, are as follows:

Policy Class	**Loss Amount**		
	0	**100**	**500**
A	0.90	0.07	0.03
B	0.50	0.30	0.20
C	0.30	0.33	0.37

(a) (*2 points*) Show that the expected value of the hypothetical means is 120 to the nearest 10. You should calculate the value to the nearest 0.1.

(b) (*2 points*) Show that the expected value of the process variance, v, is 30,700 to the nearest 100. You should calculate the value to the nearest 1.

(c) (*2 points*) Show that the variance of the hypothetical means, a, is 6400 to the nearest 100. You should calculate the value to the nearest 1.

A policy is chosen at random and is observed to have losses of 500 in the first exposure period, 100 in the second exposure period, and 100 in the third exposure period.

(d) (*2 points*) Determine the Bühlmann credibility estimate of the net premium for the policy's fourth exposure period.

(e) (*2 points*) Calculate the Bayesian estimate of the net premium for the policy's fourth exposure period.

(f) (*2 points*)

(i) Suppose that v increases while a stays the same. Explain why the resulting change in the Bühlmann credibility factor (that is, increase, decrease, or stay the same) is reasonable.

(ii) Suppose that a increases while v stays the same. Explain why the resulting change in the Bühlmann credibility factor is reasonable.

(iii) Suppose that the number of years of data increases, while a and v remain the same. Explain why the resulting change in the Bühlmann credibility factor is reasonable.

6.

(*11 points*) An actuary is investigating the severity of losses from a commercial line for reinsurance pricing, using a data set of 500 observations of loss severity, denoted *X*. The actuary first proposes to estimate the loss severity using a frailty model with frailty random variable Λ. You are given:

- Λ follows a Gamma distribution with mean α/γ and variance α/γ^2.
- Given $\Lambda = \lambda$, the conditional hazard rate of *X* is $h_{X|\Lambda}(x|\lambda) = \lambda$.

(a) (*2 points*) Show that the unconditional survival function of *X* is

$$S_X(x) = \left(\frac{\gamma}{\gamma + x}\right)^{\alpha}, \qquad x \geq 0,$$

which is the survival function for a Pareto distribution.

(b) (*3 points*) You are given that the maximum likelihood estimates (MLEs) of α and γ based on the 500 observations are $\hat{\alpha} = 4.19$ and $\hat{\gamma} = 6{,}726.8$.

(i) Show that the 97.5% Value at Risk (VaR) of the losses using the fitted frailty model for *X* is 9,500 to the nearest 100. You should calculate the value to the nearest 1.

(ii) Calculate the 97.5% Expected Shortfall (ES) of the losses using the fitted frailty model for *X*.

(c) (*2 points*) Using normalizing functions $c_n = \gamma n^{1/\alpha}$ and $d_n = -\gamma$, show that the distribution function of *X* is in the maximum domain of attraction (MDA) of the Fréchet distribution.

The actuary then decides to estimate the risk measures, VaR and ES, using the Generalized Pareto Distribution (GPD) with a threshold of $d = 8{,}900$ for the excess loss distribution. The MLEs of the GPD parameters are $\hat{\xi} = 0.85$ and $\hat{\beta} = 1{,}595.6$.

You are given the following table showing an excerpt from the largest 20 values from the sample of 500, in decreasing order.

Rank	500	499	…	484	483	482	481
Loss	38,439	22,491	…	8,916	8,402	8,342	8,110

6. **Continued**

(d) (*3 points*)

(i) Show that the 97.5% VaR of the losses using the fitted GPD is 9,500 to the nearest 100. You should calculate the value to the nearest 1.

(ii) Calculate the 97.5% ES of the losses using the fitted GPD.

(e) (*1 point*) Your colleague at your reinsurance company suggests that since the tail risk of the losses has been captured through fitting a fat-tailed frailty model, it is not necessary to perform excess loss analysis using an extreme value distribution. Critique this suggestion.

****END OF EXAMINATION****